# The Fibrifold Notation and Classification for 3D Space Groups

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# *History*

- This work was started over ten years ago by John Conway and William Thurston

- Manuscript: **Two and Three-Dimensional Orbifold Voodoo**

- Last couple of months we have redone and completed this work

- Olaf Delgado and D.H. (1997): Invariants (and software) that distinguish between all 219 different space group/orbifolds

# Plane groups

| | |
|---|---|
| $p1$ | $pm$ |
| $p2$ | $pmg$ |
| $p3$ | $pmm$ |
| $p4$ | $p31m$ |
| $p6$ | $p3m1$ |
| $cm$ | $p4g$ |
| $cmm$ | $p4m$ |
| $pg$ | $p6m$ |
| $pgg$ | |

# *Plane groups*

| | |
|---|---|
| ○ | ** |
| 2222 | 22* |
| 333 | *2222 |
| 442 | 3*3 |
| 632 | *333 |
| *× | 4*2 |
| 2*22 | *442 |
| ×× | *632 |
| 22× | |

# The orbifold notation

A 2d orbifold $\Omega = (M, S)$ consists of a compact surface $M$ (with or without a boundary) and a discrete set $S \subset M$ of points in $M$ that are labeled with *digits* $\geq 2$.
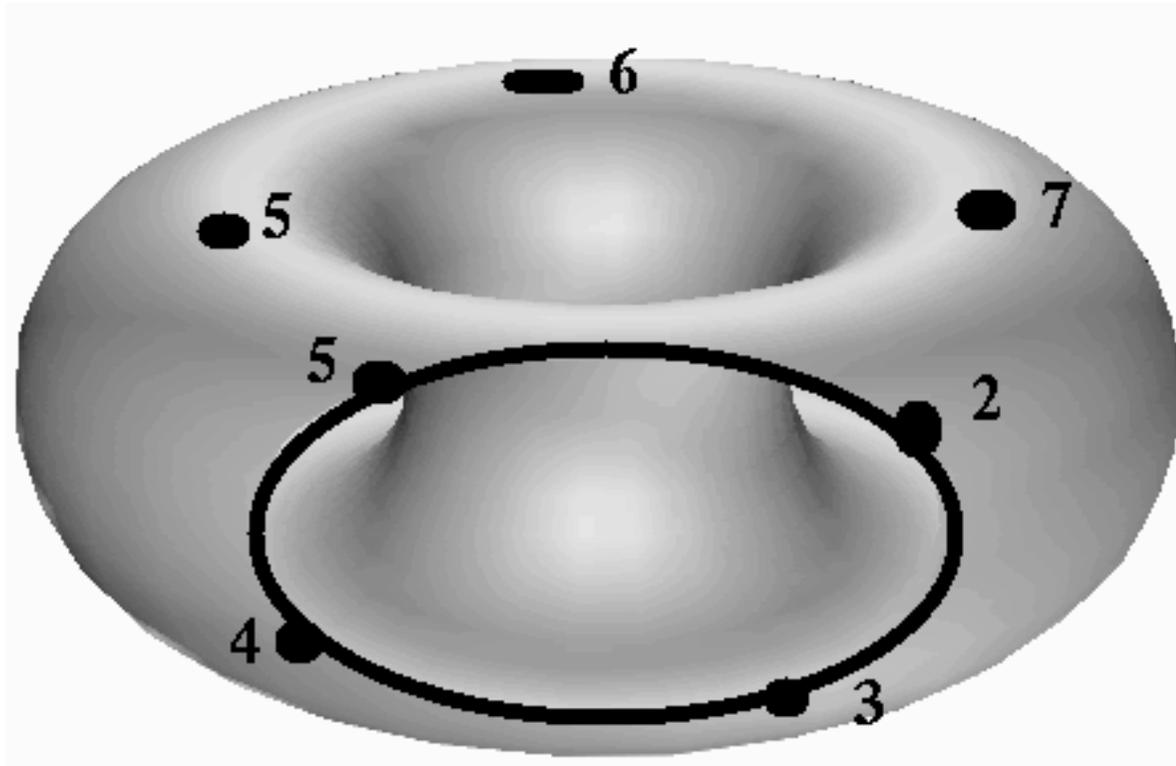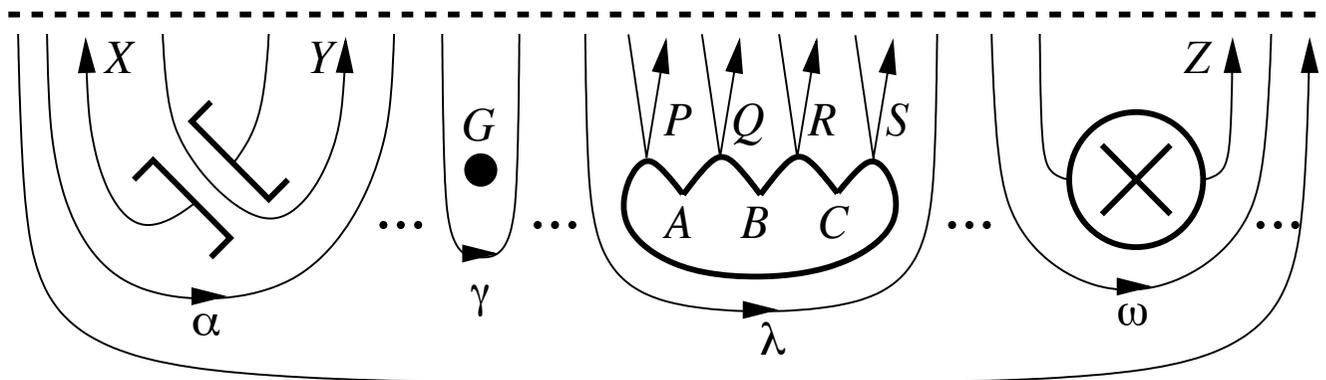


∘567∗2345

# Components of the orbifold notation

$$\underbrace{\circ\circ\ldots}_{} \qquad \underbrace{GH\ldots}_{} \qquad \underbrace{*ABC\ldots}_{} \quad \ldots \qquad \underbrace{\times\times\ldots}_{}$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$$

| handles | cone-points | corner-points | cross caps |
|---------|-------------|---------------|------------|
| translations | gyrations | kaleidoscopes | glide-reflections |
| (rings) | (large digits) | (small digits) | (crosses) |

- In the presence of $\times$, $\circ$ can be replace by $\times\times$

- Special cases:
  1 sphere, $\circ$ torus, $\times\times$ Klein bottle, $*pqr$ triangle group

# Generators and relations for two-dimensional groups

$$\alpha_{\circ}{}^{X\,Y} \;\; \beta_{\circ}{}^{X'\,Y'} \;\ldots\; \gamma_G \;\; \delta_H \ldots \; \lambda_*{}^P A^Q B^R C^S \ldots \; \mu_*{}^T D^U \ldots \; \psi_\times{}^Z \; \omega_\times{}^{Z'} \ldots.$$



$$
\begin{aligned}
\alpha_{\circ}{}^{X\,Y} \;\; &: \;\; X^{-1}Y^{-1}XY = [X, Y] = \alpha \\
\gamma_G \;\; &: \;\; \gamma^G = 1 \\
\lambda_*{}^P A^Q B \ldots {}^R C^S \;\; &: \;\;
\begin{cases}
1 = P^2 = (PQ)^A = Q^2 = (QR)^B \\
\quad = R^2 = (RS)^C = S^2, \\
\lambda^{-1}P\lambda = S
\end{cases} \\
\omega_\times{}^Z \;\; &: \;\; Z^2 = \omega \\
\text{global} \;\; &: \;\; \alpha \ldots \gamma \ldots \lambda \ldots \omega = 1
\end{aligned}
$$

# *Why use the orbifold notation?*

- Orbifold notation is "self-contained", no lookup table required

- Applies to $\mathbb{E}^2$, $\mathbb{S}^2$ and $\mathbb{H}^2$, also to infinite ribbon

- Reveals relationships between different groups (across geometries) that are obscured by other naming schemes

- Simple to distinguish names from "non-names"
  (Only non-names are: $P$, $PQ$, $*p$ and $*pq$)

# *Also possible for three-dimensional groups?*

- Two-dimensional surfaces are easily classified by number of handles or cross caps and boundary components

- A three-dimensional orbifold $\Omega$ is in general hard to describe: It consists of a three-dimensional space together with an embedded edge-labeled graph

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# A "simple" three-dimensional orbifold
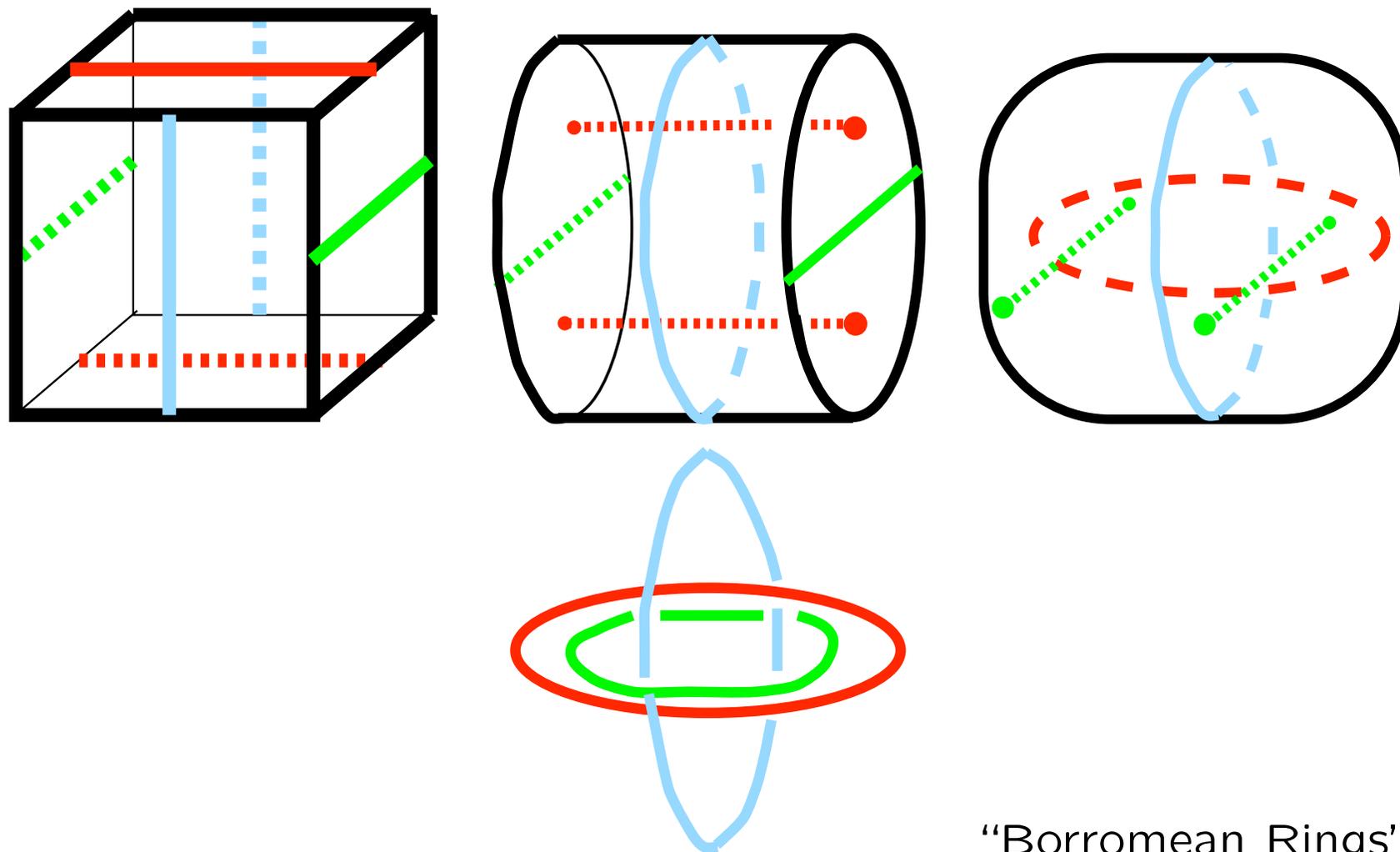


"Borromean Rings"

# Fibered space groups

Conway and Thurston's approach: consider fibered space groups

- **Fibration**: division of space into a system of parallel lines

- **Fibered space group**: space group together with an invariant fibration

- **Fibered orbifold:** Obtained on division of a fibered space by a fibered group

- **Reducible group**: possesses an invariant direction

# Example of a fibered group
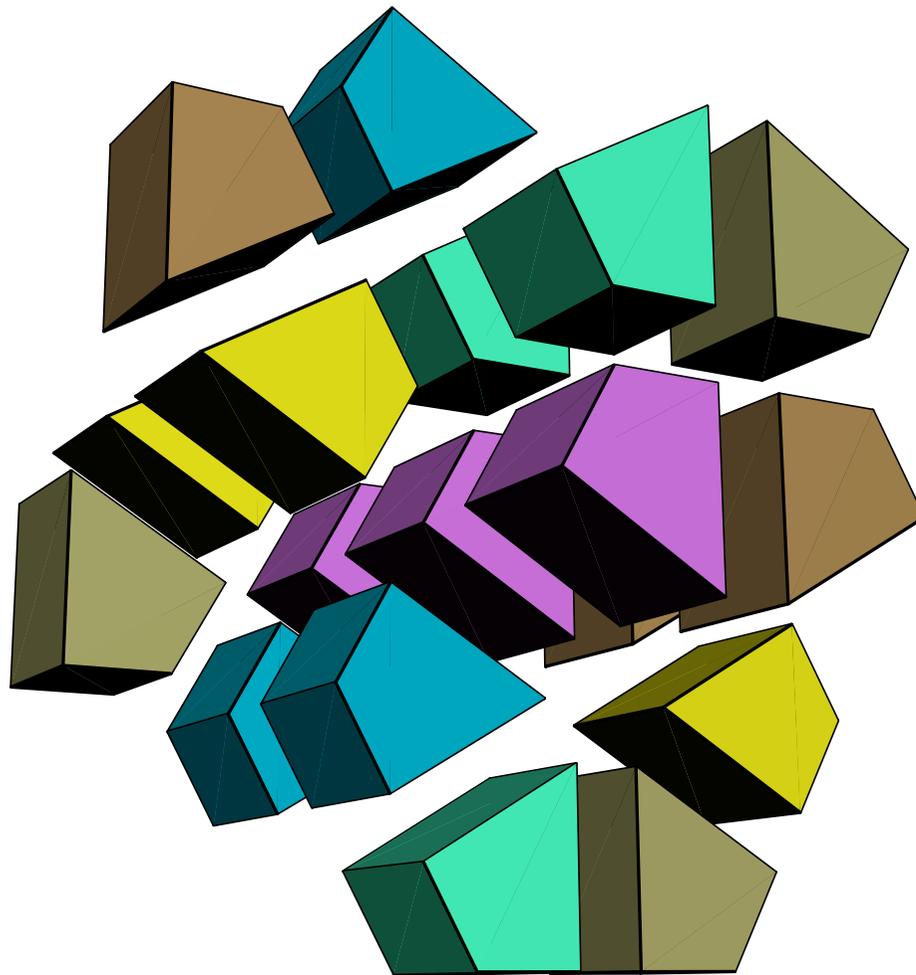
# Example of a irreducible group
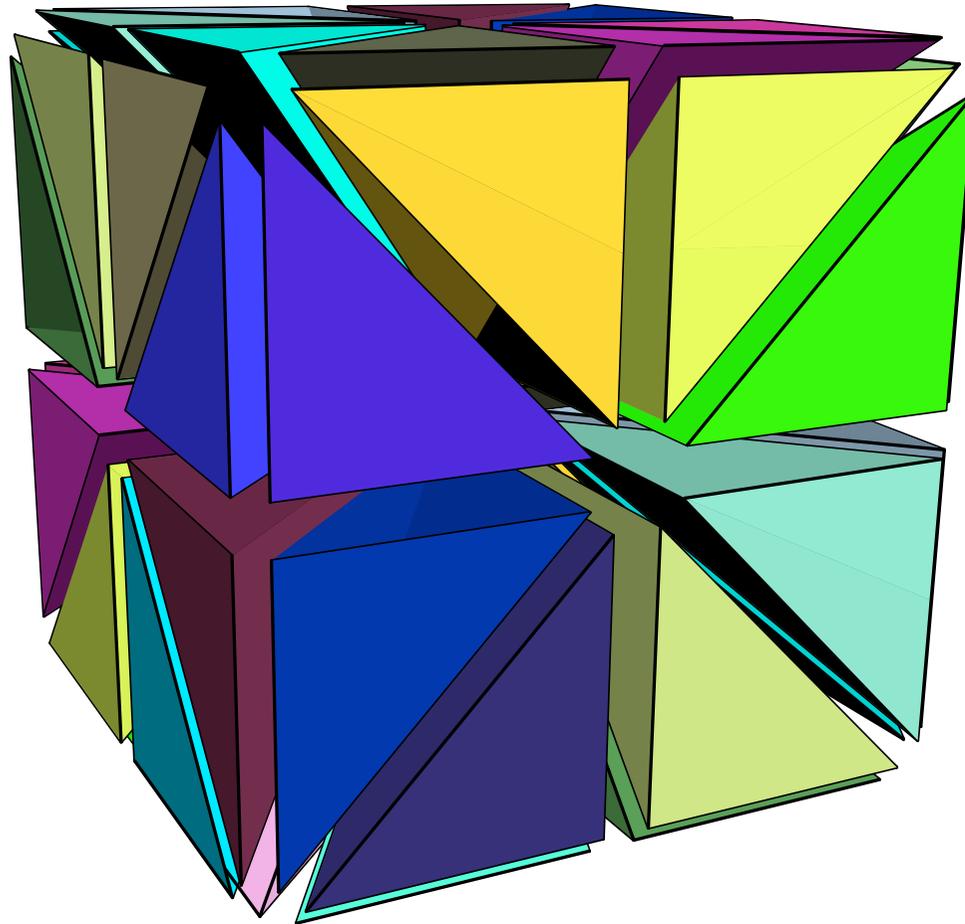
# *Horizontal and vertical operations*

Assume $G$ is fibered in $z$ direction over a plane group $H$

Every element $g \in G$ has the form:

$$g : (x, y, z) \rightarrow (\ \underbrace{a(x,y), b(x,y)}_{\uparrow}, \ \underbrace{c \pm z}_{\uparrow})$$

**horizontal**    **vertical**
operation    operation

For vertical operations we write:

$$c+ \text{ for: } z \longrightarrow c + z \qquad c- \text{ for: } z \longrightarrow c - z$$

translation through $c$        reflection in plane at height $\frac{c}{2}$

# *Specifying a fibered space group*

- Given a plane group $H = \langle P, Q, R \ldots \rangle$

- A fibered space group $G$ is obtained by assigning vertical operators $p\pm, q\pm, r\pm \ldots$ to the generators $P, Q, R \ldots$ of $H$

- The *kernel* $K$ (set of all vertical operations coupled with $I \in H$) is
  either $\qquad \{n+ \mid n \in \mathbb{N}\} \qquad$ or $\qquad \{n\pm \mid n \in \mathbb{N}\}$

  $\qquad$ Circle-fiber case $\qquad$ Interval-fiber case

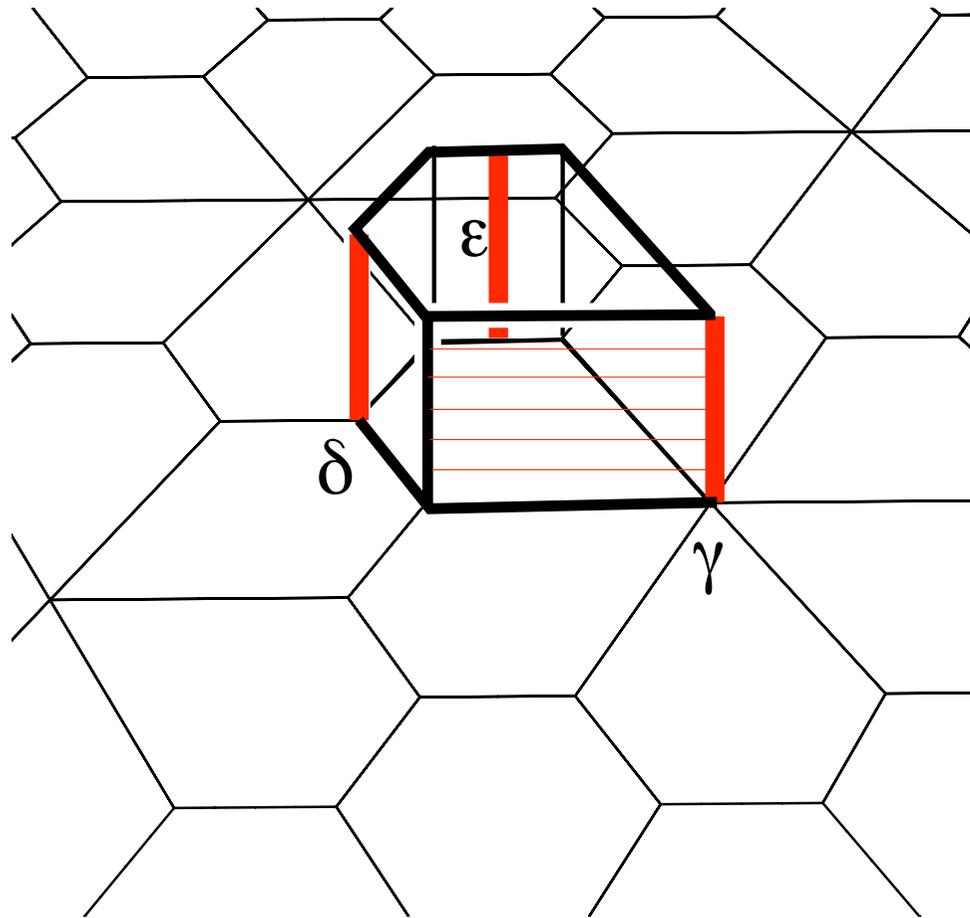- Condition: must be a homomorphism (modulo $K$)

- Need only specify $c\pm$ modulo $K$, so can assume $0 \leq c < 1$

# *Example: plane group* 632
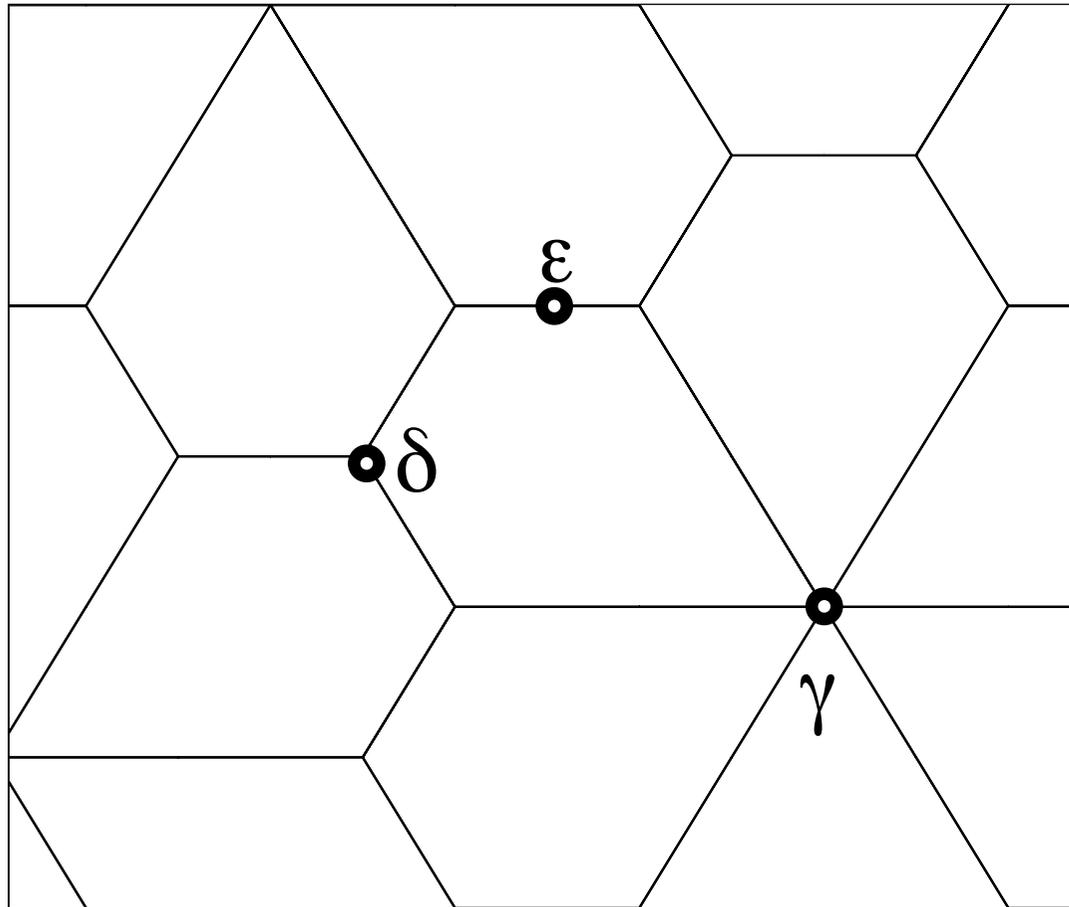


$$\gamma 6^\delta 3^\epsilon 2 \quad : \quad 1 = \gamma^6 = \delta^3 = \epsilon^2 = \gamma\delta\epsilon$$

# *Example: plane group* 632



$$\gamma 6^\delta 3^\epsilon 2 \quad : \quad 1 = \gamma^6 = \delta^3 = \epsilon^2 = \gamma\delta\epsilon$$

# *Possible couplings for* 632

- Relations $^{\gamma}6^{\delta}3^{\epsilon}2$ : $1 = \gamma^6 = \delta^3 = \epsilon^2 = \gamma\delta\epsilon$

- Determine vertical couplings: $\gamma \to c\pm$ , $\delta \to d\pm$ and $\epsilon \to e\pm$

- $\gamma\delta\epsilon = 1$: need only compute $d\pm$ and $e\pm$

  In the circle-fiber case:

- $\delta^3 = 1$: $d\pm$ is $0+$ , $\frac{1}{3}+$ , or $\frac{2}{3}+$     (odd order: $d\pm$ can't have $-$ sign)

- $\epsilon^2 = 1$: $e\pm$ is $0+$ , $\frac{1}{2}+$ , or $0-$     (if sign is $-$ , can "drive" $e$ to 0)

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# Couplings for $^\gamma 6^\delta 3^\epsilon 2$, circle-fiber case

9 cases ($\delta \in \{0+, \frac{1}{3}+, \frac{2}{3}+\}$, $\epsilon \in \{0+, \frac{1}{2}+, 0-\}$):

| $\gamma$ | $\delta$ | $\epsilon$ | | $\gamma$ | $\delta$ | $\epsilon$ | | $\gamma$ | $\delta$ | $\epsilon$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $0+$ | $0+$ | $0+$ | | $\frac{2}{3}+$ | $\frac{1}{3}+$ | $0+$ | | $\frac{1}{3}+$ | $\frac{2}{3}+$ | $0+$ |
| $\frac{1}{2}+$ | $0+$ | $\frac{1}{2}+$ | | $\frac{1}{6}+$ | $\frac{1}{3}+$ | $\frac{1}{2}+$ | | $\frac{5}{6}+$ | $\frac{2}{3}+$ | $\frac{1}{2}+$ |
| $0-$ | $0+$ | $0-$ | | $\frac{1}{3}-$ | $\frac{1}{3}+$ | $0-$ | | $\frac{2}{3}-$ | $\frac{2}{3}+$ | $0-$ |

Note: replacing the sign of $z$ reduces to six cases, all distinct

# Couplings for $^\gamma 6^\delta 3^\epsilon 2$, interval-fiber case

- Only possible couplings: $0+$ and $\frac{1}{2}+$     (because $K = \{n\pm\}$)

- $\delta^3 = 1$:     $d\pm = 0+$

- 2 cases:

$$
\begin{array}{ccc}
\gamma & \delta & \epsilon \\
0+ & 0+ & 0+
\end{array}
\qquad
\begin{array}{ccc}
\gamma & \delta & \epsilon \\
\frac{1}{2}+ & 0+ & \frac{1}{2}+
\end{array}
$$

# *Introducing the fibrifold notation...*

- Given (a presentation of) a plane group $H$, enumerating all fiberations over it is straight-forward: list all valid vertical couplings

- **Problem**: different couplings can give rise to the same fibration, e.g. due to symmetries of the plane group, but there are also much more subtle symmetries

- Need a notation that is *not* specified by the couplings themselves, but by features of the fibered orbifold (up to isotopy)

- The **Fibrifold notation** solves this problem as it is based on features of the fibration (up to isotopy), but some cases are very tricky.

# *Fibrifold names for $^\gamma 6^\delta 3^\epsilon 2$*

Circle-fiber case:

| $\gamma$ | $\delta$ | $\epsilon$ | | | | fibrifold name |
|---|---|---|---|---|---|---|
| $0+$ | $0+$ | $0+$ | | | | $(6_0 3_0 2_0)$ |
| $\frac{1}{2}+$ | $0+$ | $\frac{1}{2}+$ | | | | $(6_3 3_0 2_1)$ |
| $0-$ | $0+$ | $0-$ | | | | $(6\,3_0 2)$ |
| $\frac{2}{3}+$ | $\frac{1}{3}+$ | $0+$ | $\cong$ | $\frac{1}{3}+\ \frac{2}{3}+\ 0+$ | | $(6_2 3_2 2_0)$ |
| $\frac{1}{6}+$ | $\frac{1}{3}+$ | $\frac{1}{2}+$ | $\cong$ | $\frac{5}{6}+\ \frac{2}{3}+\ \frac{1}{2}+$ | | $(6_1 3_1 2_1)$ |
| $\frac{1}{3}-$ | $\frac{1}{3}+$ | $0-$ | $\cong$ | $\frac{2}{3}-\ \frac{2}{3}+\ 0-$ | | $(6\,3_1 2)$ |

Interval-fiber case:

| $\gamma$ | $\delta$ | $\epsilon$ | fibrifold name |
|---|---|---|---|
| $0+$ | $0+$ | $0+$ | $[6_0 3_0 2_0]$ |
| $\frac{1}{2}+$ | $0+$ | $\frac{1}{2}+$ | $[6_3 3_0 2_1]$ |

- If map is $\frac{a}{A}+$ , embellish $A$ to $A_a$ (rotation number)
- If map is $c-$ , leave $A$ unembellished (height of reflection plane not feature)

# *Fibrifold names for* $*^P 3^Q 3^R 3$



P  Q   R

| | | | |
|---|---|---|---|
| $0+$ | $0+$ | $0+$ | $(*\cdot3\cdot3\cdot3)$ |
| $\frac{1}{2}+$ | $\frac{1}{2}+$ | $\frac{1}{2}+$ | $(*:3:3:3)$ |
| $0-$ | $0-$ | $0-$ | $(*3_03_03_0)$ |
| $\frac{2}{3}-$ | $\frac{1}{3}-$ | $0-$ | $(*3_13_13_1)$ |
| $\frac{1}{3}-$ | $\frac{1}{3}-$ | $0-$ | $(*3_03_13_2)$ |
| | | | |
| $0+$ | $0+$ | $0+$ | $[*\cdot3\cdot3\cdot3]$ |
| $\frac{1}{2}+$ | $\frac{1}{2}+$ | $\frac{1}{2}+$ | $[*:3:3:3]$ |
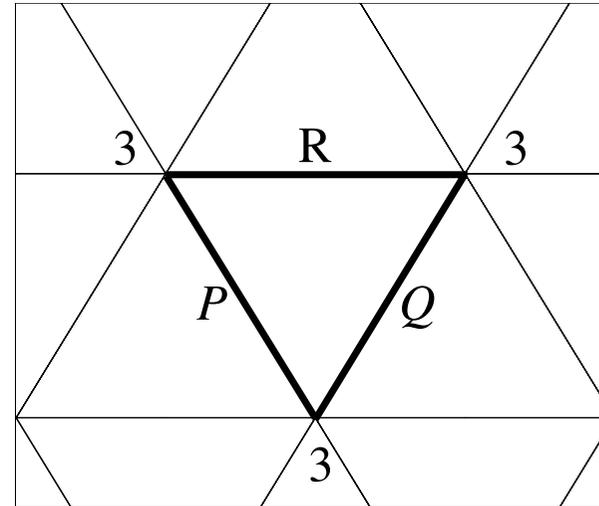
- A typical $Q$ is of order two, so $Q$ couples with $0+$ , $\frac{1}{2}+$ or $q-$
- If $Q \rightarrow 0+$ or $Q \rightarrow \frac{1}{2}+$ , write $A \cdot B$ or $A:B$
- If $P \rightarrow p-$ & $Q \rightarrow q+$ , leave $A$ unembellished (not here)
- If $P \rightarrow p-$ & $Q \rightarrow q-$ , then $PQ \rightarrow (p-q)+$ , $p-q = \frac{a}{A}$; embellish $A$ to $A_a$

(Further embellishments necessary in more subtle cases: $\bar{*}$ or $*_i$)

# *The fibrifold notation*

- Of the 219 space groups, 184 are reducible

- The 35 irreducible groups are treated separately

- Each reducible space group fibers in at most 3 ways, giving rise to *primary* and *secondary* names

- The **fibrifold notation** for $G$ over $H$ is obtained by "embellishing" the orbifold name for $H$ and describes how horizontal operations in $H$ couple to vertical operations in $G$

# *Table 1: Enumeration of fibrations*

| Plane group: $22\times$ |||||
|---|---|---|---|---|
| Relations $\,^{\gamma}2\,^{\delta}2\times^{Z}$ : |||||
| $1 = \gamma^2 = \delta^2 = \gamma\delta Z^2$ |||||
| Fibrifold name | Shifts for $\gamma \quad \delta \quad Z \quad I$ | | Point group | IT no. |
| $[2_0 2_0 \times_0]$ | $0+\ 0+\ 0+\ 0-$ | | $*222$ | 55 |
| $[2_0 2_0 \times_1]$ | $0+\ 0+\ \frac{1}{2}+\ 0-$ | | $*222$ | 58 |
| $[2_1 2_1 \times]$ | $\frac{1}{2}+\ \frac{1}{2}+\ 0+\ 0-$ | | $*222$ | 62 |
| | | | | |
| $(2_0 2_0 \times_0)$ | $0+\ 0+\ 0+$ | | $*22$ | 32 |
| $(2_0 2_0 \times_1)$ | $0+\ 0+\ \frac{1}{2}+$ | | $*22$ | 34 |
| $(2_0 2_1 \times)$ | $0+\ \frac{1}{2}+\ \frac{1}{4}+$ | | $*22$ | 43 |
| $(2_1 2_1 \times)$ | $\frac{1}{2}+\ \frac{1}{2}+\ 0+$ | | $*22$ | 33 |
| | | | | |
| $(2_0 2_0 \bar{\times})$ | $0+\ 0+\ 0-$ | | $222$ | 18 |
| $(2_1 2_1 \bar{\times})$ | $\frac{1}{2}+\ \frac{1}{2}+\ 0-$ | | $222$ | 19 |
| | | | | |
| $(2\,2\times)$ | $0-\ 0-\ 0+$ | | $2*$ | 14 |

# *Very subtle symmetry (base change)*

(Reduced) relations ${}^{\gamma}2^{\delta}2\times^{Z}$: $1 = \gamma^2 = \delta^2 = \gamma\delta Z^2$
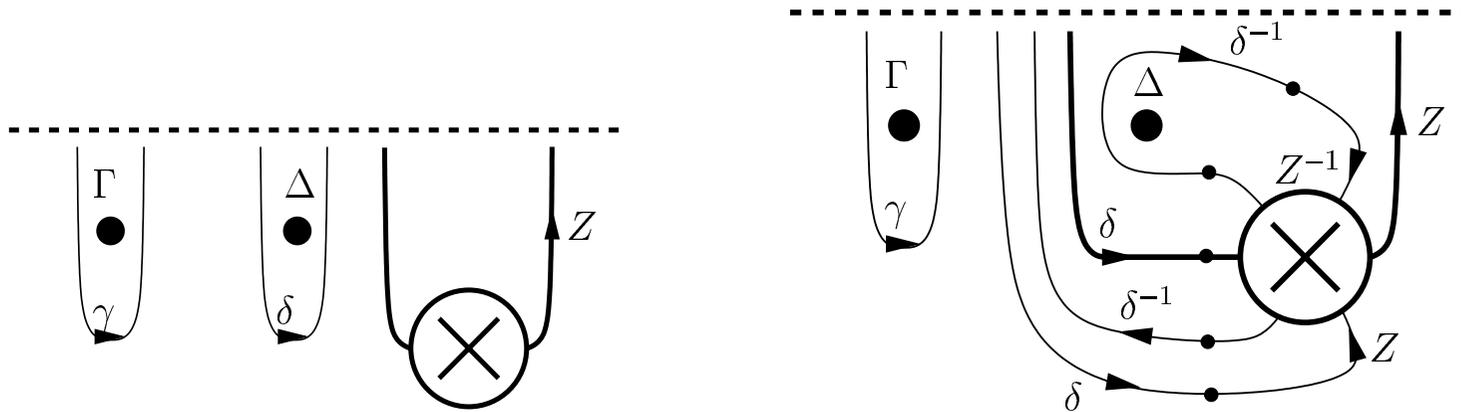


Alternative set of generators:
$$\begin{aligned}
\gamma' &= \gamma, \\
\delta' &= \delta Z \delta^{-1} Z^{-1} \delta^{-1} \\
Z' &= \delta Z
\end{aligned}$$

Reduces the list of 14 different fibrations to 10.

# *Enumeration of space groups*

- List fibrifold names by point group

- Most point groups give rise to only one invariant direction; in these cases we get one name per group

- In the other cases ($*222$, $222$, $2*$, $22$, $\times$, $*$ and $1$) can obtain up to three different names; "primary" and "secondary"

- For each IT number that has more than one fibrifold name, show that names must belong to the same group (to make classification independent of international tabulation)

# Table 2: Enumeration of space groups

| Primary name | International name and no. | Secondary names |
|---|---|---|
| Point Group $*22$,     nos. 25-46 | | |
| $(*\!\cdot\!2\!\cdot\!2\!\cdot\!2\!\cdot\!2)$ | $25.Pmm2$ | $[*_0\cdot*_0\cdot]$ |
| $(*\!\cdot\!2\!\cdot\!2\!\cdot\!2\!:\!2)$ | $38.Amm2$ | $[*_1\cdot*_1\cdot], [*\!\cdot\!\times_0]$ |
| $(*\!\cdot\!2\!\cdot\!2\!:\!2\!:\!2)$ | $42.Fmm2$ | $[*_1\cdot*_1\!:]$ |
| $(*\!\cdot\!2\!:\!2\!\cdot\!2\!:\!2)$ | $26.Pmc2_1$ | $(\overline{*}\!\cdot\!\overline{*}\cdot), [\times_0\times_0]$ |
| $(*\!\cdot\!2\!:\!2\!:\!2\!:\!2)$ | $39.Abm2$ | $[*_1\!:\!*_1\!:], (\overline{*}\!\cdot\!\overline{*}_0)$ |
| $(*\!:\!2\!:\!2\!:\!2\!:\!2)$ | $27.Pcc2$ | $(\overline{*}_0\overline{*}_0)$ |
| $(2_0*\!\cdot\!2\!\cdot\!2)$ | $35.Cmm2$ | $[*_0\cdot*_0\!:]$ |
| $(2_0*\!\cdot\!2\!:\!2)$ | $46.Ima2$ | $(\overline{*}\!\cdot\!\overline{*}_1), [*\!:\!\times_0]$ |
| $(2_0*\!:\!2\!:\!2)$ | $37.Ccc2$ | $(\overline{*}_0\overline{*}_1)$ |
| $(2_1*\!\cdot\!2\!\cdot\!2)$ | $44.Imm2$ | $[*\!\cdot\!\times_1]$ |
| $(2_1*\!\cdot\!2\!:\!2)$ | $36.Cmc2_1$ | $(\overline{*}\!\cdot\!\overline{*}\!:), [\times_1\times_1]$ |
| $(2_1*\!:\!2\!:\!2)$ | $45.Iba2$ | $(\overline{*}\!:\!\overline{*}_0)$ |

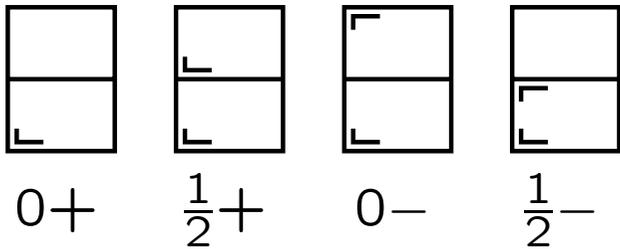| Primary name | International name and no. | Secondary names |
|---|---|---|

Rule here for choosing primary name: use canonical direction of $*22$
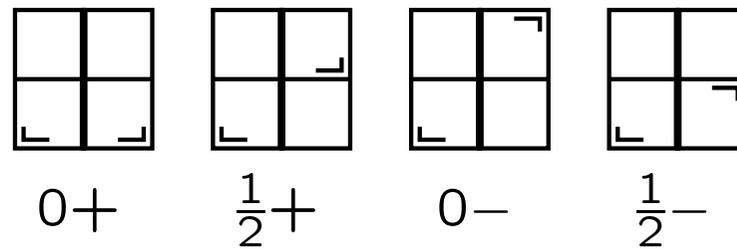
# Fundamental domain for $^\gamma 2^\delta 2 *^P$
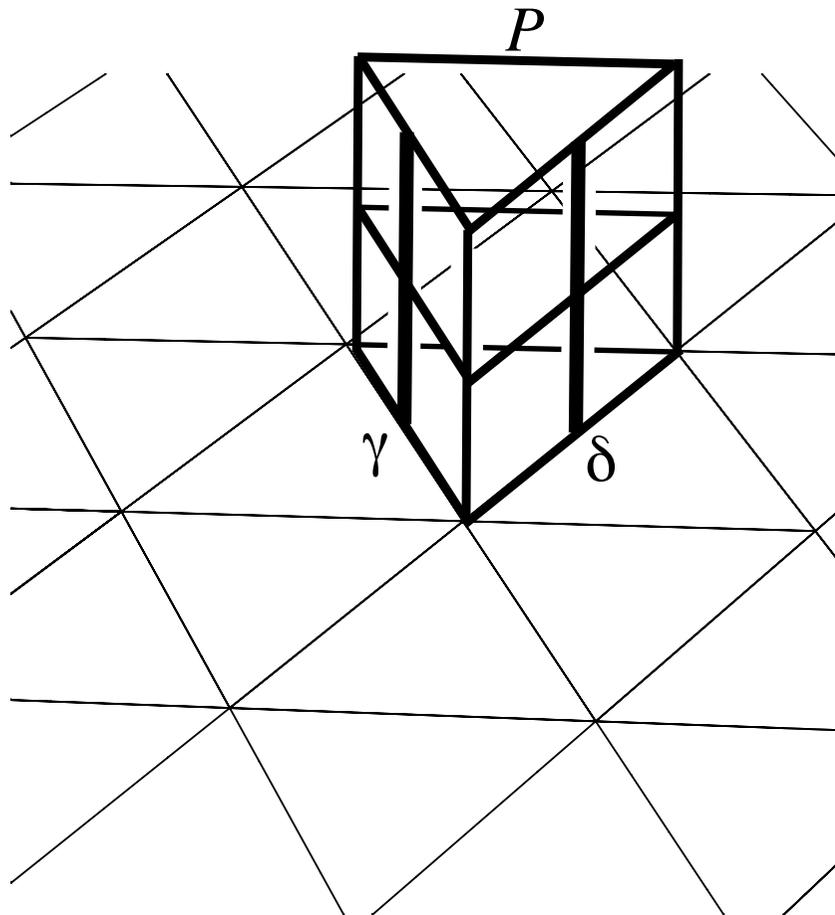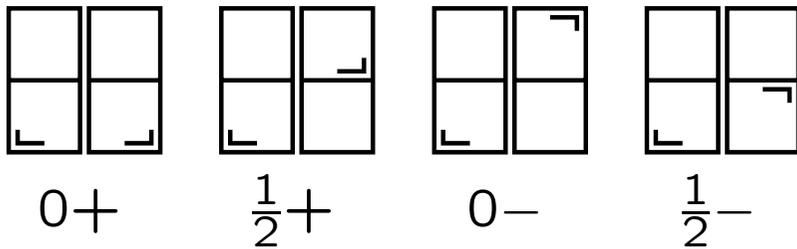
# *Geometric realization of couplings*

reflections

$0+$  $\frac{1}{2}+$  $0-$  $\frac{1}{2}-$

gyrations

$0+$  $\frac{1}{2}+$  $0-$  $\frac{1}{2}-$

translations

$0+$  $\frac{1}{2}+$  $0-$  $\frac{1}{2}-$

glide-reflections

$0+$  $\frac{1}{2}+$  $0-$  $\frac{1}{2}-$

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